ON EXISTENCE OF CONE-MAXIMAL POINTS IN REAL TOPOLOGICAL LINEAR SPACES

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ABSTRACT
A sufficient condition for a convex cone C in a Hausdorff topological linear space is given in order to ensure the existence of cone-maximal points. The condition becomes a necessary one in a topological linear space with a countable local base, that is, if the space is pseudometrizable. The paper extends known results to infinite dimensions and we answer Corley's question in the affirmative with the exception of a pathological case.

1. Introduction

In this note we study the existence of cone-maximal points in topological linear spaces in the sense described below. Throughout the paper Y is a topological linear space over reals and $C \subsetneq Y$ is a convex cone (i.e., $tC \subseteq C$ for $t \geqq 0$ and C is nonempty and convex). For $y_1, y_2 \in Y$ we write $y_1 \leqq_C y_2$ if $y_2 - y_1 \in C$. Subsequently $\leqq$ will be written instead of $\leqq_C$. The relation $\leqq$ is reflexive and since C is convex we get that $\leqq$ is also transitive, but not necessarily antisymmetric. If C is a pointed cone, i.e. $C \cap (-C) = \{0\}$, then $\leqq$ is antisymmetric.

We say that an element e of a subset B of Y is maximal up to indifference with respect to C (or in short i-maximal) in B if whenever $e \leqq y$, for some $y \in B$, then $y \leqq e$, and we write $e \in E_C(B)$ [2,5,9]. See, for instance, [10,18] for nonconical orderings.

An element e of a subset B of Y is said to be maximal in B if $\{b \in B; e \leqq b, b \neq e\} = \emptyset$, and we write $e \in e_C(B)$ [1,4,11,15,17,20].

Both definitions coincide when the cone C is pointed.

More references can be found in [19] and in the above-mentioned papers.

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Throughout the paper $\mathbf{R}$ and $\mathbf{R}^n$ denote the space of real numbers and n-dimensional Euclidean space, respectively. Moreover, $\bar{A}$ is the closure of a subset A in Y and A^c denotes complementation.

Let us note that even for a finite set $B \subseteq \mathbf{R}^2$, $e_C(B)$ may be empty. However Yu [20] proved that in $\mathbf{R}^n$, if $\bar{C}$ is pointed and B is nonempty and compact, then $e_C(B)$ ($= E_C(B)$ in this case) is nonempty. Next Hartley [9] showed that in $\mathbf{R}^n$, $E_C(B)$ is nonempty for every convex cone C and nonempty, C-compact subset B (i.e. $B \cap (y + \bar{C})$ is nonempty, compact for some $y \in Y$). Hence if B is a nonempty compact subset of $\mathbf{R}^n$ then $E_C(B) \neq \emptyset$ for every convex cone in $\mathbf{R}^n$. Corley [5] extended the result of Yu to infinite dimensions, namely he proved that if Y is a real topological linear space and $\bar{C}$ is a pointed convex cone then $e_C(B)$ ($= E_C(B)$ in this case) is nonempty for every nonempty C-semicompact set B, where B is said to be C-semicompact if every open cover of B of the form $\{(\bar{C} + y_i)^c : y_i \in B, i \in I\}$ has a finite subcover. Corley has asked the question, whether Hartley's result can be extended to infinite dimensions, namely:

Is it true that $E_C(B) \neq \emptyset$ for every convex cone C in a real topological space Y, whenever B is nonempty and compact?

We answer this question in the affirmative with an exception of a rather peculiar case (Theorem 2.2). All results cited above follow easily from Theorem 2.2.

Many authors have studied the existence of maximal points in order to relax the requirement of compactness on B. In every case stronger conditions have to be imposed on the cone. For instance, in [4] it is shown that if C is a closed convex cone in a Banach space Y, satisfying the (π)-property (i.e., there exists a continuous functional f on Y with $f(y) \leqq 0$ for each $y \in C$ such that for every $\varepsilon > 0$ the set $\{y \in C : f(y) \geqq -\varepsilon\}$, if nonempty, is relatively weakly compact in Y) then $e_C(B)$ ($= E_C(B)$ in this case) is nonempty for every nonempty, weakly closed and bounded above set B in Y. Another result in $\mathbf{R}^n$ can be found in [1,11,12], to name only a few.

Moreover, observe that even for a closed, convex, pointed cone C in a Banach space, $E_C(B)$ ($= e_C(B)$ in this case) may be empty for a closed and bounded set B; and simple examples in $\mathbf{R}^2$ show that $E_C(B)$ may be empty for a compact set B if C is closed and pointed but not convex. Both examples are given in [5].

2. Results

We shall need the following theorem, proved by Borwein [3, Theorem 1c] which is the next generalization of existence results in infinite dimensions. Let us note here that the same conclusion is obtained in [10, Theorem 2.3, Corollary 2.8] for a preorder relation (i.e., reflexive and transitive binary relation) on a normed space Y, with closed upper sections.

THEOREM 2.1. *Let Y be a topological linear space, and C be a closed, convex cone in Y. Then $E_C(B)$ is nonempty whenever a subset B of Y is nonempty and compact.*

If C is a convex cone then by $\operatorname{lin} C$ we denote the maximal linear subspace contained in C, i.e., $\operatorname{lin} C = C \cap (-C)$. If C is a closed convex cone then $\operatorname{lin} C$ is a closed subspace.

Now we are going to prove that the crucial assumption on a convex cone C in a topological linear space Y, for $E_C(B)$ to be nonempty, for any nonempty and compact subset B of Y, is:

(*) For every closed subspace L of $\operatorname{lin} \bar{C}$, if $\overline{C \cap L}$ is a linear subspace then so is $C \cap L$.

The above condition is equivalent to an apparently stronger statement in which we demand that the condition (*) is satisfied for any closed subspace L in Y. Indeed, if L is a closed subspace in Y such that $\overline{C \cap L}$ is linear, then $\overline{C \cap L}$ is a closed subspace of $\operatorname{lin} \bar{C}$. Moreover $C \cap \overline{C \cap L} = C \cap L$. Using the condition (*), $C \cap L = C \cap \overline{C \cap L}$ must be a linear subspace as well.

REMARK 2.1. The condition (*) is fulfilled in any finite-dimensional space Y for every convex cone C.

PROOF. Using the separation theorem in finite-dimensional spaces [12, p. 15] we get that $C \cap L = \overline{C \cap L}$. ■

REMARK 2.2. The condition (*) is satisfied in any real topological space Y if:
(i) C is a closed convex cone;
(ii) $\bar{C}$ is a pointed, convex cone;
(iii) $C \setminus \{0\}$ is an open, convex cone.

PROOF. (i) and (ii) are obvious.
To prove (iii) let us assume that L is a closed subspace of Y such that $\overline{C \cap L}$ is linear. As $C' = C \setminus \{0\}$ is open and convex we get that $C' \cap L$ is an open convex cone in $\overline{C \cap L}$. Applying the separation theorem [12, p. 63] to the subset $C' \cap L$ in $\overline{C \cap L}$ we get that $C \cap L = \overline{C \cap L}$. ■

The next example shows that in any infinite-dimensional normed space Y

there exists a convex cone C with int $C \neq \emptyset$, which does not satisfy the condition (*).

EXAMPLE 2.1. Let Y be an infinite-dimensional normed space and $f: Y \to \mathbf{R}$ be a continuous linear functional. Put $C_1 = \{0\} \cup \{y \in Y : f(y) < 0\}$. As $\ker f = \{y \in Y : f(y) = 0\}$ is an infinite dimensional, closed subspace of Y, we can find a bounded sequence $\{e_n\}_{n=1}^{\infty}$ consisting of linear independent elements in $\ker f$, and the closed subspace L generated by $\{e_n\}_{n=1}^{\infty}$, i.e. $L = \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}}$, is contained in $\ker f$. Define $y_n = e_1 + (n+1)^{-1} e_{n+1}$ for $n = 1, 2, \ldots$ and the convex cone

$$C_2 = \left\{ \sum_{n=1}^{k} t_n y_n + t e_1 : t_n \in \mathbf{R},\ n = 1, 2, \ldots, k,\ t \leqq 0,\ k \text{ is a natural number} \right\} \subseteq L.$$

Then C_2 is not a linear space, since $-e_1 \in C_2$ and $e_1 \notin C_2$, but $\bar{C}_2 = L$. Consider $C = C_1 + C_2$ which is a convex cone and $\emptyset \neq \operatorname{int} C_1 \subseteq \operatorname{int} C$. It is easy to check that $C \cap L = C_2$, hence C does not satisfy the condition (*). ■

Let us note that we can give the same construction as in Example 2.1 in any topological linear space Y with nontrivial, linear, continuous functionals, which admits a bounded subset contained in no finite-dimensional linear subspace of Y. One may also observe that every infinite-dimensional space Y contains a pointed convex cone C such that $\bar{C} = Y$ (see [12, p. 10], [14, p. 454]).

THEOREM 2.2. *Let Y be a Hausdorff topological linear space and C be a convex cone satisfying the condition (*). Then for every nonempty, compact set B, $E_C(B)$ is nonempty.*

PROOF. Assume that B is a nonempty compact set in Y. The proof will go by transfinite induction. If C is a linear space then obviously $E_C(B) = B \neq \emptyset$. Assume that C is not a linear space. Hence $\bar{C}$ is also not a linear space by the assumption (*). We define the closed subspace $L_1 = \operatorname{lin} \bar{C} \subsetneq Y$, the convex cone $C_1 = C \cap L_1$ and by Theorem 2.1 we have that there exists $e_1 \in E_{\bar{C}}(B)$. If C_1 is a linear space we finish at this stage since $e_1 \in E_C(B)$ as well. If C_1 is not a linear space, then by the assumption (*) $\bar{C}_1$ cannot be linear too, and we define the nonempty compact set $B_1 = B \cap (L_1 + e_1)$, the subspace $L_2 = \operatorname{lin} \bar{C}_1 \subsetneq L_1$, the convex cone $C_2 = C_1 \cap L_2$ and by Theorem 2.1 we get an element $e_2 \in E_{\bar{C}_1}(B_1)$.

In general, let α be an arbitrary ordinal number. Assume that we have defined the subspaces L_β, convex cones $C_\beta \subseteq L_\beta$, nonempty compact sets B_β and elements $e_\beta \in B$, for every $\beta < \alpha$.

I. $\alpha - 1$ exists.

(i) If the cone $C_{\alpha-1}$ is a linear space we finish at the $\alpha - 1$ stage.

(ii) If the cone $C_{\alpha-1}$ is not a linear space then by the assumption (*) $\overline{C_{\alpha-1}}$ is not a linear space as well and we define the α-stage as follows:

the space $L_\alpha = \operatorname{lin} \overline{C_{\alpha-1}} \subsetneq L_{\alpha-1}$,

the convex cone $C_\alpha = C_{\alpha-1} \cap L_\alpha$,

an arbitrary element $e_\alpha \in E_{\overline{C_{\alpha-1}}}(B_{\alpha-1})$, which exists by Theorem 2.1,

the nonempty compact set $B_\alpha = B_{\alpha-1} \cap (L_\alpha + e_\alpha)$.

II. α is a limit ordinal.

(i) If for some $\beta < \alpha$, C_β is a linear space, we do not construct the α-stage.

(ii) If for every $\beta < \alpha$, C_β is not a linear space, then we define:

the subspace $L_\alpha = \bigcap_{\beta<\alpha} L_\beta$,

the convex cone $C_\alpha = C \cap \bigcap_{\beta<\alpha} L_\beta (= \bigcap_{\beta<\alpha} C_\beta)$,

the nonempty compact set $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$, as $\{B_\beta\}_{\beta<\alpha}$ is a decreasing family of nonempty compact sets,

the element e_α is arbitrary in $E_{\bar{C}_\alpha}(B_\alpha)$, which is nonempty by Theorem 2.1.

We first show for every ordinal number α such that $\alpha - 1$ exists that we have $E_{C_\alpha}(B_\alpha) \subseteq E_{C_{\alpha-1}}(B_{\alpha-1})$. We shall show first that $E_{C_\alpha}(B_\alpha) \subseteq E_{C_{\alpha-1}}(B_\alpha)$. Indeed, if $x \in E_{C_\alpha}(B_\alpha)$ and $b - x \in C_{\alpha-1}$ for some $b \in B_\alpha$ then both x and b belong to B_α, which means that $x, b \in L_\alpha + e_\alpha$. Hence $b - x \in L_\alpha \cap C_{\alpha-1} = C_\alpha$, which implies that $x - b \in C_\alpha \subseteq C_{\alpha-1}$, thus $x \in E_{C_{\alpha-1}}(B_\alpha)$.

Now we show that $E_{C_{\alpha-1}}(B_\alpha) \subseteq E_{C_{\alpha-1}}(B_{\alpha-1})$. Let $x \in E_{C_{\alpha-1}}(B_\alpha)$ and $b - x \in C_{\alpha-1}$ for some $b \in B_{\alpha-1}$. Then $x = l_\alpha + e_\alpha$ for some $l_\alpha \in L_\alpha$ and $b - l_\alpha - e_\alpha = b - x \in C_{\alpha-1}$. Hence $b - e_\alpha \in C_{\alpha-1} + L_\alpha = C_{\alpha-1} + \operatorname{lin} \overline{C_{\alpha-1}} \subseteq \overline{C_{\alpha-1}}$. Since by the construction, $e_\alpha \in E_{\overline{C_{\alpha-1}}}(B_{\alpha-1})$ and $b \in B_{\alpha-1}$ we must have that $e_\alpha - b \in \overline{C_{\alpha-1}}$. Thus we get that $b - e_\alpha \in \operatorname{lin} \overline{C_{\alpha-1}} = L_\alpha$ and $b \in B_{\alpha-1} \cap (L_\alpha + e_\alpha) = B_\alpha$. As $x \in E_{C_{\alpha-1}}(B_\alpha)$ we obtain that $x - b \in C_{\alpha-1}$, which means that $x \in E_{C_{\alpha-1}}(B_{\alpha-1})$.

If α is a limit ordinal then $E_{C_\alpha}(B_\alpha) \subseteq E_C(B)$. Indeed, let $x \in E_{C_\alpha}(B_\alpha)$ and $b - x \in C$ for some $b \in B$. We shall show by transfinite induction that $b \in B_\beta$ and $b - x \in C \cap L_\beta$ for every $\beta < \alpha$. As $x \in B_\alpha = \bigcap_{\beta<\alpha} B_\beta$ we get that $x \in B_1 = B \cap (L_1 + e_1)$ and $x = l_1 + e_1$ for some $l_1 \in L_1$. Then $b - l_1 - e_1 = b - x \in C$ and $b - e_1 \in C + L_1 = C + \operatorname{lin} \bar{C} \subseteq \bar{C}$. By the construction $e_1 \in E_{\bar{C}}(B)$ so we must have that $e_1 - b \in \bar{C}$, which implies $b - e_1 \in \operatorname{lin} \bar{C} = L_1$. Hence $b \in B \cap (L_1 + e_1) = B_1$ and $b - x = b - l_1 - e_1 \in L_1 + L_1 = L_1$, thus $b - x \in C \cap L_1$.

Fix $\beta < \alpha$ and assume that our hypothesis is true for $\gamma < \beta$, i.e. $b \in B_\gamma$ and $b - x \in C \cap L_\gamma$ for every $\gamma < \beta$. There are again two cases:

(a) $\beta - 1$ exists.

In this case $b \in B_{\beta-1}$ and $b - x \in C \cap L_{\beta-1}$. As $x \in B_\alpha \subseteq L_\beta + e_\beta$ we have that

$x = l_\beta + e_\beta$ for some $l_\beta \in L_\beta$ and $b - l_\beta - e_\beta = b - x \in C \cap L_{\beta-1} = C_{\beta-1}$. Hence

$$b - e_\beta \in C_{\beta-1} + L_\beta = C_{\beta-1} + \operatorname{lin} \overline{C_{\beta-1}} \subseteq \overline{C_{\beta-1}}.$$

As $e_\beta \in E_{\overline{C_{\beta-1}}}(B_{\beta-1})$, $b \in B_{\beta-1}$ we must have that $e_\beta - b \in \overline{C_{\beta-1}}$, which implies that $b - e_\beta \in \operatorname{lin} \overline{C_{\beta-1}} = L_\beta$. Hence $b \in B_{\beta-1} \cap (L_\beta + e_\beta) = B_\beta$ and $b - x = b - l_\beta - e_\beta \in L_\beta + L_\beta$ which means that $b - x \in C \cap L_\beta$.

(b) β is a limit ordinal. By the definition of B_β in this case we have that $B_\beta = B \cap \bigcap_{\gamma<\beta}(L_\gamma + e_\gamma) = \bigcap_{\gamma<\beta} B_\gamma$, so if $b \in B_\gamma$ for every $\gamma < \beta$ then $b \in B_\beta$ as well. Moreover $L_\beta = \bigcap_{\gamma<\beta} L_\gamma$, so $b - x \in C \cap \bigcap_{\gamma<\beta} L_\gamma = C \cap L_\beta$.

(a) and (b) prove that $b \in B_\beta$ and $b - x \in C \cap L_\beta$ for every $\beta < \alpha$. Hence $b \in \bigcap_{\beta<\alpha} B_\beta = B_\alpha$ and $b - x \in C \cap \bigcap_{\beta<\alpha} L_\beta = C_\alpha$ and therefore we must have $x - b \in C_\alpha \subseteq C$, which means that $x \in E_C(B)$ and completes the proof of the fact that $E_{C_\alpha}(B_\alpha) \subseteq E_C(B)$ if α is a limit ordinal.

By the above consideration we can see that $E_{C_\alpha}(B_\alpha) \subseteq E_C(B)$ for every ordinal α. Therefore in order to prove that $E_C(B) \neq \emptyset$ it is sufficient to show that there exists α with $E_{C_\alpha}(B_\alpha) \neq \emptyset$. However our transfinite sequence of the subspaces must stop (see Remark 2.3); when it does, it can only be because the corresponding cone C_α is a linear subspace and in this case $\emptyset \neq B_\alpha = E_{C_\alpha}(B)$. ■

Remark 2.3. The result that the process of constructing the transfinite sequence of linear subspaces must stop follows from the Axiom of Replacement in the presence of the other axioms of Zermelo–Fraenkel set theory, if one uses the von Neumann definition of the ordinal numbers [8]. This means that we do not use the Axiom of Choice here. However, we construct the transfinite sequence of elements, $e_\alpha \in B$ for α an ordinal number, by Theorem 2.1, which depends upon the Axiom of Choice. Precisely, Borwein [3, Theorem 7] proved that the Axiom of Choice is equivalent to the existence of maximal points for compact, convex sets.

The next result is a converse to Theorem 2.2 for topological linear spaces with the First Axiom of Countability, which is equivalent to pseudometrizability [13, p. 48]. The question of whether the condition (∗) is necessary for $E_C(B)$ to be nonempty, for every nonempty and compact subset B, in non-pseudometrizable linear spaces remains open.

Proposition 2.1. *Let Y be a topological linear space and let C be a convex cone in Y, which does not fulfil the condition (∗). Suppose that* $\operatorname{lin} \bar{C}$ *satisfies the First Axiom of Countability, then there exists a nonempty compact set B in Y with* $E_C(B) = \emptyset$.

PROOF. As C does not satisfy (*) we can find a closed subspace L in $\operatorname{lin}\bar{C}$ such that $\overline{C \cap L}$ is a linear space but $C \cap L$ is not linear. Observe that $E_C(B) \subseteq E_{C\cap L}(B)$ for every set B. Hence it is sufficient to prove that $E_{C\cap L}(B) = \emptyset$ for some nonempty and compact set B. Without loss of generality we may assume that $Y = L$ and $C \cap L = C$. Thus we get that $\bar{C}$ is a linear space while C is not and Y is first countable.

First we shall show that for every $x \in C$ such that $-x \notin C$ there exists a sequence $(x_n)_{n=1}^{\infty}$ tending to $-x$ with $x_n \in C$, $-x_n \notin C$ for $n = 1,2,\dots$. As $\bar{C}$ is a linear space, and Y satisfies the I Axiom of Countability we get that $-x \in \bar{C}$ if $x \in C$ and there exists a sequence $(c_n)_{n=1}^{\infty}$ converging to $-x$ such that $c_n \in C$ for $n = 1,2,\dots$. Define $x_n = (1/n)x + (1-1/n)c_n$ for $n = 1,2,\dots$. Since C is convex we get that $x_n \in C$ and $-x_n \notin C$ for $n = 1,2,\dots$. Moreover the sequence $(x_n)_{n=1}^{\infty}$ tends to $-x$. Hence we obtain that for every $x \in C$ such that $-x \notin C$ and for every neighbourhood of zero U we can find $y \in C$ so that $x + y \in U$ and $-y \notin C$.

Since C is not a linear space, we can find $y_0 \in C$ so that $-y_0 \notin C$. Let $(U_n)_{n=1}^{\infty}$ be a base of neighbourhoods of 0. Let $y_1 \in C$ be such that $y_0 + y_1 \in U_1$ and $-y_1 \notin C$. Then $y_0 + y_1 \in C$ and $-(y_0 + y_1) \notin C$, since $-y_0 \notin C$. Inductively we determine a sequence $(y_n)_{n=1}^{\infty}$ for which $y_0 + \sum_{i=1}^{n} y_i \in U_n$, $y_n \in C$ and $-y_n \notin C$ for $n = 1,2,\dots$. Then $\sum_{n=1}^{\infty} y_n = -y_0$. Put

$$B = \{-y_0\} \cup \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^{n} y_i \right\}.$$

Then B is nonempty and compact, but $-y_0 \leqq \sum_{i=1}^{n} y_i$, $\sum_{i=1}^{n} y_i \not\leqq -y_0$, $\sum_{i=1}^{n} y_i \leqq \sum_{i=1}^{n+1} y_i$ and $\sum_{i=1}^{n+1} y_i \not\leqq \sum_{i=1}^{n} y_i$ for $n = 1,2,\dots$, which implies that $E_C(B) = \emptyset$.

COROLLARY 2.1. *Let Y be a Hausdorff topological linear space, C a convex cone in Y satisfying the condition* (*). *If B is nonempty, C-compact subset of Y, i.e. $B \cap (y + \bar{C})$ is nonempty compact for some $y \in Y$, then $E_C(B) \neq \emptyset$.*

The proof follows from Theorem 2.2 since $E_C((y + \bar{C}) \cap B) \subseteq E_C(B)$. ■

Hartley's result [9], that in $\mathbf{R}^n$ $E_C(B) \neq \emptyset$ for every nonempty, C-compact set B follows from Remark 2.1 and Corollary 2.1.

Using Remark 2.2 (ii) and Corollary 2.1 we get that $E_C(B) \neq \emptyset$ for every nonempty C-compact (hence for compact as well) set B, whenever $\bar{C}$ is pointed, in any Hausdorff topological space Y. Moreover by Corollary 2.1 and Remark 2.2 (iii) we obtain that the set of so-called weak or quasi-maximals [6,7,15,16] is nonempty if B is nonempty and C-compact. This is the case when $y_1 \leqq y_2$ means $y_2 - y_1 \in \operatorname{int} C$ for $y_1, y_2 \in Y$, where $\operatorname{int} C$ denotes the interior of C.

COROLLARY 2.2. *Let Y be a Hausdorff topological linear space, C be a convex cone and M be a linear supspace of Y such that $M \cap \operatorname{lin} \bar{C}$ is finite dimensional. Then for every nonempty, C-compact subset B of M, $E_C(B) \neq \varnothing$.*

PROOF. If L is a closed subspace of M such that $\overline{C \cap M \cap L} = \overline{C \cap L}$ is a linear space, then $\overline{C \cap L}$ must be finite dimensional, since then $\overline{C \cap L} \subseteq M \cap \operatorname{lin} \bar{C}$. Hence $C \cap L = \overline{C \cap L}$ is a linear space, which proves that $C \cap M$ admits the condition (*) on M. As $E_{C \cap M}(A) \subseteq E_C(A)$ for any subset A of M, we get by Corollary 2.1 that $E_C(B) \neq \varnothing$. ■

COROLLARY 2.3. *Let Y be a Hausdorff topological linear space and C be a convex cone. Then for every non-empty, C-compact subset B of Y, such that $(\operatorname{span} B) \cap \operatorname{lin} \bar{C}$ is finite dimensional, $E_C(B) \neq \varnothing$.*

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